Frequency response: Resonance, Bandwidth, Q factor

Resonance.

Let's continue the exploration of the frequency response of RLC circuits by investigating the series RLC circuit shown on Figure 1.

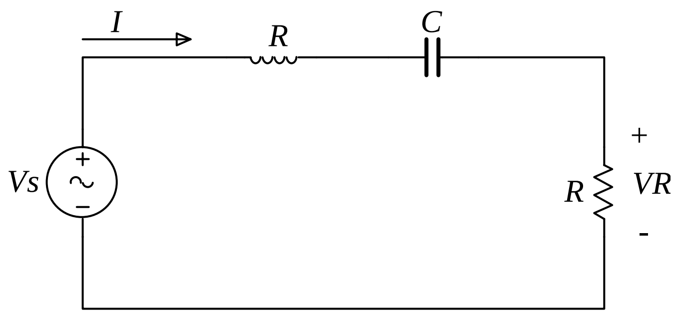


Figure 1

The magnitude of the transfer function when the output is taken across the resistor is

$$|H(\omega)| \equiv \left|\frac{VR}{Vs}\right| = \frac{\omega RC}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}} \tag{1.1}$$

At the frequency for which the term $1-\omega^2 LC = 0$ the magnitude becomes

$$|H(\omega)| = 1 \tag{1.2}$$

The dependence of $|H(\omega)|$ on frequency is shown on Figure 2 for which $L=47mH$ and $C=47\mu F$ and for various values of R.

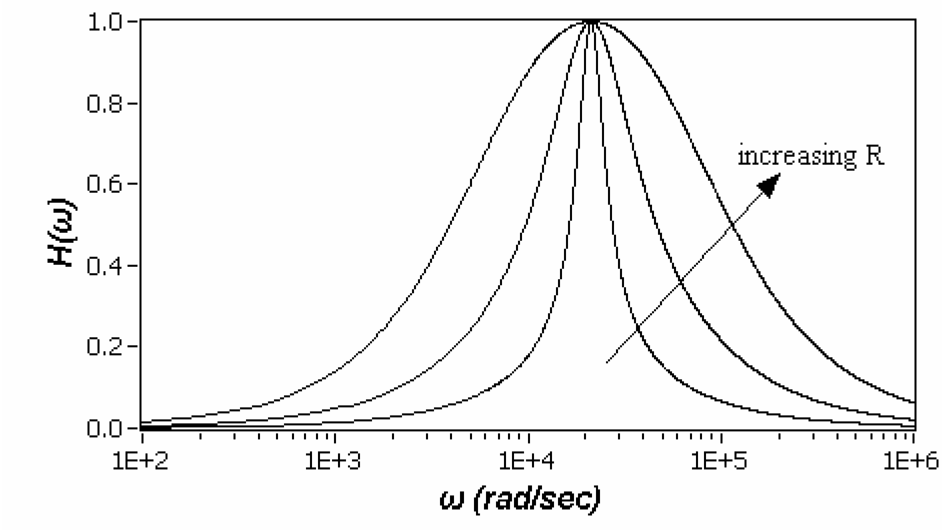


Figure 2.

The frequency $\omega_0 = \dfrac{1}{\sqrt{LC}}$ is called the resonance frequency of the RLC network.

The impedance seen by the source *Vs* is

$$\begin{aligned} Z &= R + j\omega L + \frac{1}{j\omega C} \\ &= R + j\left(\omega L - \frac{1}{\omega C}\right) \end{aligned} \tag{1.3}$$

Which at $\omega = \omega_0 = \dfrac{1}{\sqrt{LC}}$ becomes equal to R.

- **Therefore at the resonant frequency the impedance seen by the source is purely resistive.**
- **This implies that at resonance the inductor/capacitor combination acts as a short circuit.**
- **The current flowing in the system is in phase with the source voltage.**

The power dissipated in the RLC circuit is equal to the power dissipated by the resistor. Since the voltage across a resistor $\left(V_R \cos(\omega t)\right)$ and the current through it $\left(I_R \cos(\omega t)\right)$ are in phase, the power is

$$\begin{aligned} p(t) &= V_R \cos(\omega t) I_R \cos(\omega t) \\ &= V_R I_R \cos^2(\omega t) \end{aligned} \tag{1.4}$$

And the average power becomes

$$P(\omega) = \frac{1}{2} V_R I_R$$
$$= \frac{1}{2} I_R^2 R \tag{1.5}$$

Notice that this power is a function of frequency since the amplitudes V_R and I_R are frequency dependent quantities.

The maximum power is dissipated at the resonance frequency

$$P_{\max} = P_{(\omega=\omega_0)} = \frac{1}{2}\frac{V_S^2}{R} \tag{1.6}$$

Bandwidth.

At a certain frequency the power dissipated by the resistor is half of the maximum power which as mentioned occurs at $\omega_0 = \frac{1}{\sqrt{LC}}$. The half power occurs at the frequencies for which the amplitude of the voltage across the resistor becomes equal to $\frac{1}{\sqrt{2}}$ of the maximum.

$$P_{1/2} = \frac{1}{4}\frac{V_{max}^2}{R} \tag{1.7}$$

Figure 3 shows in graphical form the various frequencies of interest.

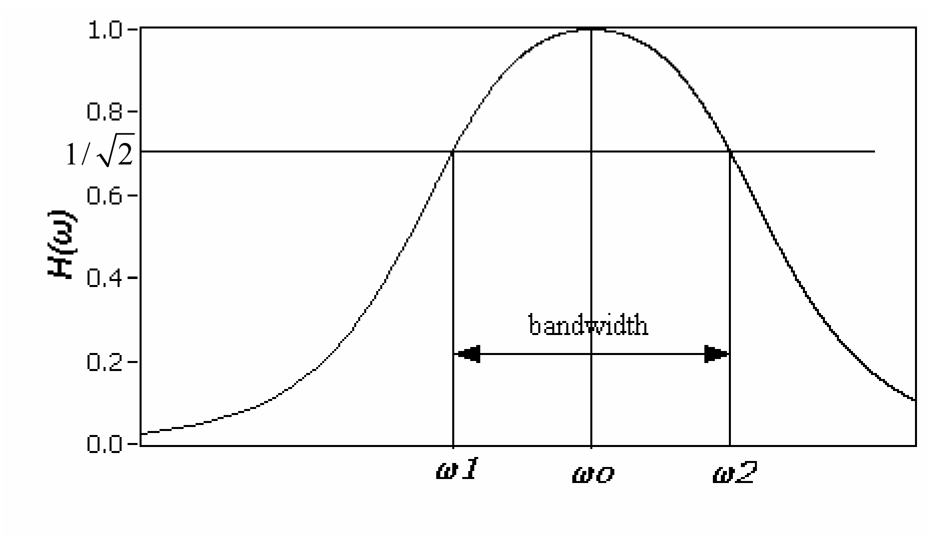


Figure 3

Therefore, the ½ power occurs at the frequencies for which

$$\frac{1}{\sqrt{2}} = \frac{\omega RC}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}} \tag{1.8}$$

Equation (1.8) has two roots

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{\omega_0^2}} \tag{1.9}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{\omega_0^2}} \tag{1.10}$$

The bandwidth is the difference between the half power frequencies

$$Bandwidth = B = \omega_2 - \omega_1 \tag{1.11}$$

By multiplying Equation (1.9) with Equation (1.10) we can show that ω_0 is the geometric mean of ω_1 and ω_2.

$$\omega_0 = \sqrt{\omega_1 \omega_2} \tag{1.12}$$

As we see from the plot on Figure 2 the bandwidth increases with increasing *R*. Equivalently the sharpness of the resonance increases with decreasing *R*.

For a fixed L and C, a decrease in R corresponds to a narrower resonance and thus a higher selectivity regarding the frequency range that can be passed by the circuit.

As we increase R, the frequency range over which the dissipative characteristics dominate the behavior of the circuit increases. In order to quantify this behavior we define a parameter called the ***Quality Factor Q*** which is related to the sharpness of the peak and it is given by

$$Q = 2\pi \frac{\text{maximum energy stored}}{\text{total energy lost per cycle at resonance}} = 2\pi \frac{E_S}{E_D} \tag{1.13}$$

which represents the ratio of the energy stored to the energy dissipated in a circuit.

The energy stored in the circuit is

$$E_S = \frac{1}{2} L I^2 + \frac{1}{2} C Vc^2 \tag{1.14}$$

For $Vc = A\sin(\omega t)$ the current flowing in the circuit is $I = C\frac{dVc}{dt} = \omega CA\cos(\omega t)$. The total energy stored in the reactive elements is

$$E_S = \frac{1}{2} L\omega^2 C^2 A^2 \cos^2(\omega t) + \frac{1}{2} CA^2 \sin^2(\omega t) \tag{1.15}$$

At the resonance frequency where $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$ the energy stored in the circuit becomes

$$E_S = \frac{1}{2} CA^2 \tag{1.16}$$

The energy dissipated per period is equal to the average resistive power dissipated times the oscillation period.

$$E_D = R\left\langle I^2 \right\rangle \frac{2\pi}{\omega_0} = R\left(\frac{\omega_0^2 C^2 A^2}{2}\right)\frac{2\pi}{\omega_0} = 2\pi\left(\frac{1}{2}\frac{RC}{\omega_0 L}A^2\right) \qquad (1.17)$$

And so the ratio Q becomes

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC} \qquad (1.18)$$

- **The quality factor increases with decreasing R**
- **The bandwidth decreases with decreasing R**

By combining Equations (1.9), (1.10), (1.11) and (1.18) we obtain the relationship between the bandwidth and the Q factor.

$$B = \frac{L}{R} = \frac{\omega_0}{Q} \qquad (1.19)$$

Therefore:

A band pass filter becomes more selective (small B) as Q increases.

Similarly we may calculate the resonance characteristics of the parallel RLC circuit.

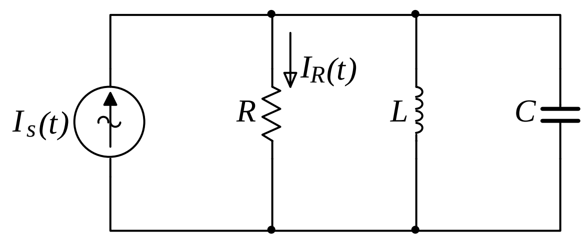


Figure 4

Here the impedance seen by the current source is

$$Z_{//} = \frac{j\omega L}{(1-\omega^2 LC) + \dfrac{j\omega L}{R}} \tag{1.20}$$

At the resonance frequency $1-\omega^2 LC = 0$ and the impedance seen by the source is purely resistive. The parallel combination of the capacitor and the inductor act as an open circuit. Therefore at the resonance the total current flows through the resistor.

If we look at the current flowing through the resistor as a function of frequency we obtain according to the current divider rule

$$\begin{aligned} I_R &= I_S \frac{\dfrac{1}{Z_R}}{\dfrac{1}{Z_R}+\dfrac{1}{Z_C}+\dfrac{1}{Z_L}} \\ &= I_S \frac{j\omega L}{(R-\omega^2 LCR) + j\omega L} \end{aligned} \tag{1.21}$$

And the transfer function becomes

$$|H(\omega)| = \left|\frac{I_R}{I_S}\right| = \frac{\omega L}{\sqrt{\left(R-\omega^2 LCR\right)^2 + \left(\omega L\right)^2}} \tag{1.22}$$

Again for *L=47mH* and *C=47μF* and for various values of *R* the transfer function is plotted on Figure 5.

For the parallel circuit the half power frequencies are found by letting $|H(\omega)| = \frac{1}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}} = \frac{\omega L}{\sqrt{\left(R - \omega^2 LCR\right)^2 + \left(\omega L\right)^2}} \tag{1.23}$$

Solving Equation (1.23) for ω we obtain the two ½ power frequencies.

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{\omega_0^2}} \tag{1.24}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{\omega_0^2}} \tag{1.25}$$

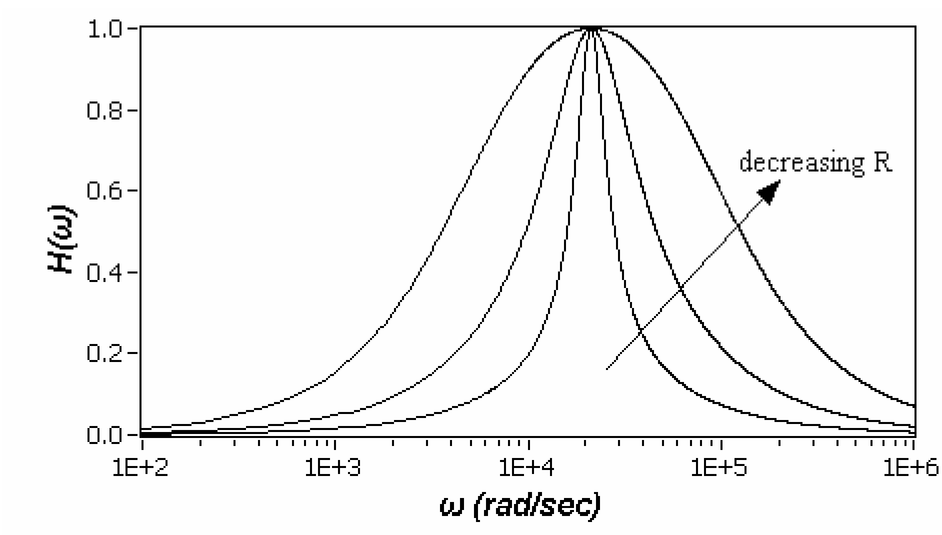


Figure 5

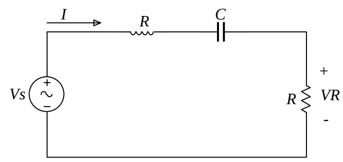
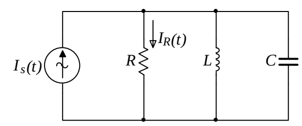
And the bandwidth for the parallel RLC circuit is

$$B_P = \omega_2 - \omega_1 = \frac{1}{RC} \tag{1.26}$$

The Q factor is

$$Q = \frac{\omega_0}{B_P} = \omega_0 RC = \frac{R}{\omega_0 L} \tag{1.27}$$

Summary of the properties of *RLC* resonant circuits.

	Series	**Parallel**
Circuit		
Transfer function	$\lvert H(\omega)\rvert \equiv \left\lvert\frac{VR}{Vs}\right\rvert = \frac{\omega RC}{\sqrt{\left(1-\omega^2 LC\right)^2+\left(\omega RC\right)^2}}$	$\lvert H(\omega)\rvert = \left\lvert\frac{I_R}{I_S}\right\rvert = \frac{\omega L}{\sqrt{\left(R-\omega^2 LCR\right)^2+\left(\omega L\right)^2}}$
Resonant frequency	$\omega_0 = \frac{1}{\sqrt{LC}}$	$\omega_0 = \frac{1}{\sqrt{LC}}$
½ power frequencies	$\omega_1 = -\frac{R}{2L}+\sqrt{\left(\frac{R}{2L}\right)^2+\frac{1}{\omega_0^2}}$ $\omega_2 = \frac{R}{2L}+\sqrt{\left(\frac{R}{2L}\right)^2+\frac{1}{\omega_0^2}}$	$\omega_1 = -\frac{1}{2RC}+\sqrt{\left(\frac{1}{2RC}\right)^2+\frac{1}{\omega_0^2}}$ $\omega_2 = \frac{1}{2RC}+\sqrt{\left(\frac{1}{2RC}\right)^2+\frac{1}{\omega_0^2}}$
Bandwidth	$B_S = \omega_2-\omega_1 = \frac{R}{L}$	$B_P = \omega_2-\omega_1 = \frac{1}{RC}$
Q factor	$Q = \frac{\omega_0}{B_S} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$	$Q = \frac{\omega_0}{B_P} = \omega_0 RC = \frac{R}{\omega_0 L}$

Example:

A very useful circuit for rejecting noise at a certain frequency such as the interference due to 60 Hz line power is the band reject filter sown below.

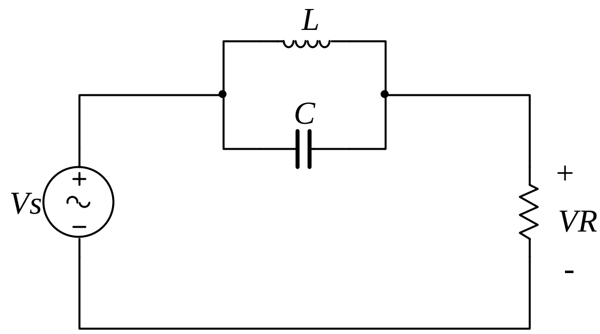


Figure 6

The impedance seen by the source is

$$Z = R + \frac{j\omega L}{1-\omega^2 LC} \tag{1.28}$$

When $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$ the impedance becomes infinite. The LC combination resembles an open circuit.

If we take the output across the resistor the magnitude of the transfer function is

$$\left|\frac{VR}{Vs}\right| = |H(\omega)| = \frac{R(1-\omega^2 LC)}{\sqrt{\left(R - R\omega^2 LC\right)^2 + \left(\omega L\right)^2}} \tag{1.29}$$

Consideration of the frequency limits gives

$$\begin{aligned} &\omega = 0, && |H(\omega)| = 1 \\ &\omega = \omega_{0,} && |H(\omega)| = 0 \\ &\omega \to \infty, && |H(\omega)| \to 1 \end{aligned} \tag{1.30}$$

which is a band-stop “notch” filter.

If we are interested in suppressing a 60 Hz noise signal then

$$2\pi\, 60 = \frac{1}{\sqrt{LC}} \tag{1.31}$$

For $L=47mH$, the corresponding value of the capacitor is $C=150\mu F$.

The plot of the transfer function with the above values for L and C is shown on Figure 7 for various values of R.

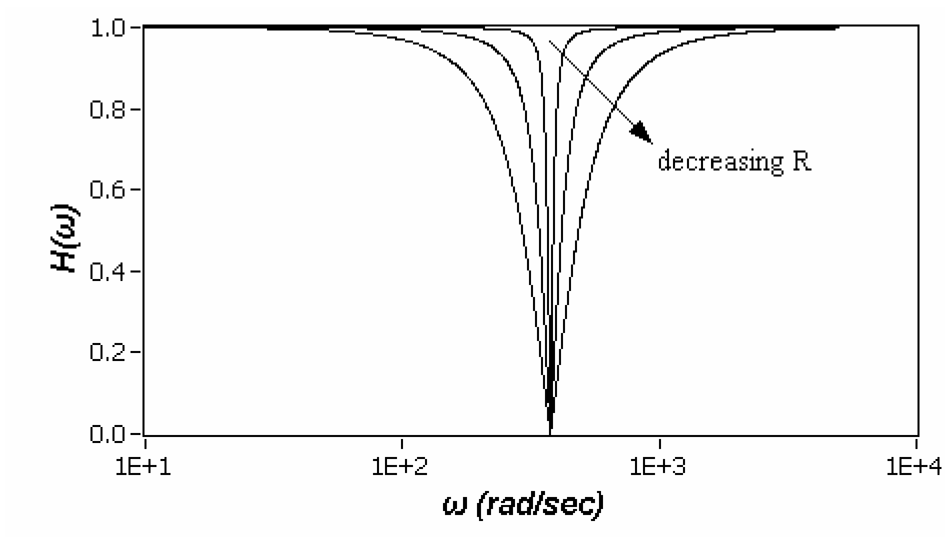


Figure 7

Since the capacitor and the inductor are in parallel the bandwidth for this circuit is

$$B = \frac{1}{RC} \tag{1.32}$$

If we require a bandwidth of 5 Hz, the resistor R=212Ω. In this case the pot of the transfer function is shown on Figure 8.

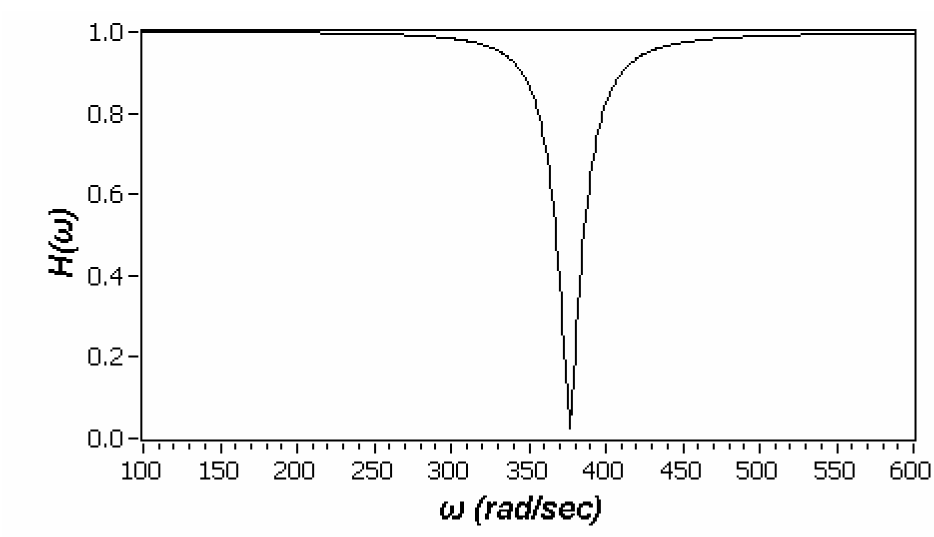


Figure 8